Necessary and Sufficient Conditions

There is a clear distinction between conditions that are necessary and/or sufficient.

- A plant needs water, but water alone may not be sufficient for growth.
- A lawyer believes an insanity plea would save his client but hopes it will not be necessary.
- A doctor realizes an operation may be necessary but fears it may not be sufficient.

It is one thing to say that something, A, is sufficient for something else, B, and another thing to say that A is necessary for B.

The sentences used to state necessary and sufficient conditions do not always contain the words ‘necessary’ or ‘sufficient’.

Example #1

a. A poor economy is enough to drive a man to drink.

A poor economy, then, is sufficient to drive a man to drink.

This would be notated as: \( E \rightarrow D \)

Whenever something (A) is said to be a sufficient condition for another thing (B), the sentence expressing this relationship will have the form \( A \rightarrow B \).

The arrow is always directed from the sufficient condition towards what it is a sufficient condition for.

Example #2

b. To please Melissa, it is sufficient to speak kindly of Brian.

Speaking kindly of Brian is all you need to do – sufficient – to please Melissa. In logical notation: \( B \rightarrow M \)

*Speaking kindly of Brian is certainly not the only way to please Melissa. It is not necessary to speak kindly of Brian to please her.

If A is a sufficient condition, it is by no means thereby a necessary condition.

Sufficient conditions are often indicated by words or phrases such as: sufficient, enough, all you have to do, you have only to, etc.

Necessary conditions are often indicated by words like ‘necessary’, ‘must’, ‘have to’, and ‘need’.
Example #1

a. You must pass the driver’s test before you are granted a driver’s license.

Passing the driver’s test is a necessary condition for being granted a driver’s license; however, this may not be a sufficient condition for there may be other prerequisites.

Example #1 would therefore be symbolized as $L \rightarrow T$: If you were granted a driver’s license, then you must have passed the driver’s test.

Example #2

b. You have to pay your library fines in order to check out more materials.

Paying your library fines is necessary to check out other materials, but it is certainly not sufficient.

Example #2 would thus be notated as $M \rightarrow F$: If you were able to check out more materials, then you had to have paid your library fines.

To say that $A$ is a necessary condition for $B$ is to say that if $B$ then $A$: $(B \rightarrow A)$. The arrow is directed toward the necessary condition and away from what it is a necessary condition for.

On occasion we want to say that $A$ is both a necessary and a sufficient condition for $B$.

Example: The existence of matter is a necessary and a sufficient condition for the existence of space. $(M, S)$

The general case for necessary and sufficient conditions taken separately apply here. Symbolized together the notation would be: $(M \rightarrow S) \cdot (S \rightarrow M)$

The more convenient notation would be: $M \leftrightarrow S$

Another way of saying the example sentence would be: There is matter if and only if there is space.

Certain common words and/or phrases are used to indicate necessary and/or sufficient conditions. Some pose difficulties in the way they affect the logical form of sentences in which they occur.

Example A

1. He will fail unless he studies. $(F, S)$ What is the relationship between the two atomic sentences controlled by the word unless?
2. $S \rightarrow \sim F$ (If he studies, then he will not fail.) or
3. $F \rightarrow \sim S$ (If he failed, then he did not study.) or
4. $\sim S \rightarrow F$ (If he did not study, then he failed.) or
5. $F \vee S$ (Either he fails or he studies.)
Examples 2 and 3 are suspect because they rule out the possibility, left open by 1 that he might study yet fail. 1 does not guarantee passing – it says only that if he does pass, he will have studied, or that if he does not study, he fails.

Thus 4 is a correct translation, as is 5. If we remember that ‘or’ means ‘at least one and perhaps both’, 5 leaves open the possibility that he may both fail and study.

6. The sentence “Either he fails or he studies.” is equivalent to example 5.

The use of ‘unless’ will result in the following equivalencies:

   He will fail unless he studies.
1. If he did not fail, then he studied.  (~F → S)
2. If he did not study, then he failed.  (~S → F)
3. Either he fails or he studies.  (F v S)
4. Either he studies or he fails.  (S v F)

Example B

1. Unless they build the dam, they will not have enough water.

Three correct symbolizations would be:

2. D v ~W (Either they build the dam, or they will not have enough water.)
3. ~D → ~W (If they do not build the dam, they will not have enough water.)
4. W → D (If they have enough water, they built the dam.)

It would be incorrect to suggest, however,

5. D → W (If they build the dam, they will have enough water.)

   • Correct translations of the form A unless B include:  A v B, ~A → B, ~B → A
   • Two more words whose translations would take the form(s) for unless would be except and until.

Another sentence link which needs to be explained is the expression ‘only if’.

Consider:

1. He will be at the races only if he has money.  (R, M)

Should this be notated as R → M (If he is at the races, then he has money.)? OR M → R (If he has money, then he is at the races.)?

All the example sentence says is having money is the only possible condition that would allow him to be at the races.

The rule one should follow when translating sentences with ‘only if’ is to regard the sentence introduced by ‘only if’ as the consequent rather than as the condition.
Sometimes like sounding sentences may be thought to have the same meaning. For instance, compare the following sentences:
1. He will neither take the train nor will he fly.
2. He will not both take the train and fly.
3. He will either take the train or he will fly.

Example 1 should be rephrased as:
He will not take the train and he will not fly. \(\neg T \cdot \neg F\)

Example 2 only denies that the two things will happen together.
\(\neg (T \cdot F)\)

Example 3 would be symbolized as: \(T \lor F\)

*The importance of correct notation is so that translating sentences into symbols maintains their meaning accurately.*

**Form and Content in Arguments**

Sentences may differ from each other in content and yet have the same form.

Example:
1. Greece’s rejection of the EU mandate will result in a financial crisis. \(G \rightarrow C\)
2. If you take the medicine, you will soon be well. \(M \rightarrow W\)
3. Were I more wealthy, I would create an endowment. \(W \rightarrow E\)

*The identity of form is shown by an arrow flanked by letters.*

To facilitate the discussion of logical forms in sentences and arguments, logicians have adopted the following notational convention:
- Let the letters ‘P,’ ‘Q,’ and ‘R’ stand for any sentences whatever.
- The letters ‘P,’ ‘Q,’ and ‘R’ will not be abbreviations for particular sentences in the way ‘G,’ ‘C,’ ‘M,’ ‘W,’ and ‘E’ are in the previous example.

For in one case ‘P’ can be:
- Greece rejects the EU mandate.

In another
- You take the medicine.

And in another
- I am more wealthy.

The three sentences symbolized previously as \(G \rightarrow C\), \(M \rightarrow W\), \(W \rightarrow E\), have the form \(P \rightarrow Q\).
The following three sentences:
1. Although he lost his goat, he is still happy.
2. It rained all summer and snowed all winter.
3. He left the courtroom, but he talked to no one.

Which would be symbolized respectively as \( L \cdot H, R \cdot S, L \cdot T \), have the form \( P \cdot Q \).

Likewise, other groups of sentences could be shown to have the forms \( P \lor Q \), \( P \rightarrow Q \), and \( \sim P \).

Arguments may differ in content but have the same form.

Example:

Jennifer either voted for Cameron, or she is against medicare. \( C \lor A \)

She did not vote for Cameron. \( \sim C \)
Therefore, she is against medicare. \( A \)

Either Rod McKuen is America’s best-loved poet, or Robert Frost is. \( M \lor F \)
Rod McKuen is not America’s best-loved poet. \( \sim M \)
Therefore Robert Frost is America’s best-loved poet. \( F \)

Texas or Alaska is the largest state in the union. \( T \lor A \)
Texas is not the largest state in the union. \( \sim T \)
Therefore, Alaska is the largest state in the union. \( A \)

While each of the previous arguments is on a different subject, they all have the same form:

\[ P \lor Q \]
\[ \sim P \]
\[ Q \]

This is a valid argument form, and for historical reasons it is called the \textbf{Disjunctive Syllogism} (D. S.)

Another frequently used valid argument form is the \textbf{Hypothetical Syllogism} (H.S.)

Example:

If the arsonists are caught, there will be a jury trial. \( A \rightarrow T \)
If there is a jury trial, the fire captain will have to testify. \( T \rightarrow C \)
Thus, if the arsonists are caught, the fire captain will have to testify. \( A \rightarrow C \)
If the solution turns red litmus paper blue, then it is alkaline. \( B \rightarrow A \)

If it is alkaline, it will have to be discarded. \( A \rightarrow D \)

Thus, if it turns litmus paper blue, it will have to be discarded. \( B \rightarrow D \)

The general form of such arguments is:

\[
\begin{align*}
P & \rightarrow Q \\
Q & \rightarrow R \\
P & \rightarrow R \quad \text{(H. S.)}
\end{align*}
\]

Another valid argument form is the **Modus Tollens**, ‘the way of denial.’ This form is frequently found in arguments of philosophers, scientists, preachers, et. al.

**Example:**

Had the husband returned before midnight, his wife would have heard him. \( R \rightarrow W \)

But his wife did not hear him. \( \sim W \)

Hence he could not have returned before midnight. \( \sim R \)

If the Labor party takes over both houses of parliament, then the present administration is unpopular. \( L \rightarrow U \)

But the present administration is not unpopular. \( \sim U \)

Therefore the Labor party will not take over both houses. \( \sim L \)

The general form of the **Modus Tollens** argument is:

\[
\begin{align*}
P & \rightarrow Q \\
\sim Q & \\
\sim P
\end{align*}
\]

**More Forms of Valid Argument**

The fact that there exists a ‘way of denial’ suggests that there might also be a ‘way of assertion’. There is.

**Modus Ponens**

One might claim that if a certain state of affairs is the case, then a certain other state of affairs must also be the case.

If this claim is accepted as true, then a demonstration that the first state of affairs does indeed exist will force acceptance of the assertion that the second state of affairs will be the case.
Suppose one is able to assert that if it rains during the harvest, the crop will mildew. Further suppose that the person with whom one is arguing accepts this assertion as true.

If one is able to demonstrate that it did indeed rain during the harvest, this forces acceptance of the consequent state of affairs – the crop will mildew.

*Put more formally and into logical notation:*

If it rains during the harvest, the crop will mildew. \(R \rightarrow M\)

It did rain during the harvest. \(R\)

Therefore, the crop will mildew. \(M\)

The **Modus Ponens** form of argument is symbolized as follows:

\[
\begin{align*}
P & \rightarrow Q \\
Q \quad \text{(M.P.)}
\end{align*}
\]

*It is possible for an extended argument to make more than one use of Modus Ponens.* For example,

If it rains, the crop will mildew. And if the crop mildews, we will lose money. It is raining, and therefore we will lose money.

The argument would be symbolized:

1. \(R \rightarrow M\)
2. \(M \rightarrow L\)
3. \(R\)
4. \(M\) (M.P. 1, 3)

C \(L\) (M.P. 2, 4)

We cannot justify the assertion of ‘L’ as immediately following from any sequence of these three premises. But if we take the first and third premise together, we are entitled to assert ‘M’.

1. \(R \rightarrow M\)
2. \(M \rightarrow L\)
3. \(R\)
4. \(M\) (M.P. 1, 3)

If we now take the second premise and step 4 together, we are entitled to assert the conclusion ‘L’ as step 5.

1. \(R \rightarrow M\)
2. \(M \rightarrow L\)
3. \(R\)
4. \(M\) (M.P. 1, 3)
C \(L\) (M.P. 2, 4)
As you probably already noticed, there is the opportunity in the previous problem to use Hypothetical Syllogism. If the first two premises are considered together, we can apply H.S. to produce ‘R→L’ as step 4.

1. \( R \rightarrow M \)
2. \( M \rightarrow L \)
3. \( R \)
4. \( R \rightarrow L \) (H.S. 1, 2)

If we now combine step 4 with step 3, Modus Ponens allows us to assert the conclusion ‘L’ as step 5.

1. \( R \rightarrow M \)
2. \( M \rightarrow L \)
3. \( R \)
4. \( R \rightarrow L \) (H.S. 1, 2)
C   L   (M.P. 4, 3)

The next valid argument form involves transposing the sentence elements of a conditional sentence. The sentence elements cannot be switched without seriously altering the meaning of the original sentence.

*The following two sentences are different in meaning:*
1. If we have run out of gas, the car will stop.
2. If the car stops, we have run out of gas.

*The problem is to find a way to transpose the order of the original simple sentences in a form that will do no harm to the conditional relationship expressed.*

**Consider:**
3. If the car does not stop, we have not run out of gas.
4. If we have not run out of gas, the car will not stop.

While ‘1’ and ‘2’ do not say the same thing, examples ‘1’ and ‘3’ seem to say the same thing.

The order of the two elements of a conditional sentence – antecedent and consequent – can be transposed by reversing their order and negating both elements.

We represent the Transposition rule (Trans.) in the following way:

\[
\begin{align*}
P \rightarrow Q \\
\sim Q \rightarrow \sim P
\end{align*}
\]
The Transposition rule is an *equivalence rule* – one which states that one form in which an assertion is made is equivalent to another. The twin vertical lines indicate that the expression may be read from top to bottom as well as from bottom to top.

The next valid argument form involves rewriting conditionals as disjunctions, and vice versa. This form is called *Implication* (Imp.).

A pastry chef might say to himself: If I’m to finish making my cake, I will have to run to the store. \((F \rightarrow R)\)

From this we can infer that: Either he does not finish making his cake, or he will have to run to the store. \((\neg F \lor R)\)

Or a student might say to himself: Either I don’t go home this weekend, or I miss the party Saturday. \((\neg H \lor M)\)

From this he can infer that: If I do go home, then I miss the party Saturday. \((H \rightarrow M)\)

The general form for the rule of Implication is as follows:

\[ P \rightarrow Q \]

\[ \neg P \lor Q \text{ (Imp.)} \]

*It is important to note that it is not only atomic sentences that can replace ‘P’, ‘Q’, and ‘R’ but also molecular sentences.*

The following premise, for example, is actually a conditional of the \(P \rightarrow Q\) form:

*If either the bond issue fails to pass or school enrollments greatly exceed the prediction, then some of our schools will need to go on double sessions, and a number of desirable programs will have to be curtailed.* \((B \lor E) \rightarrow (D \land C)\)